Locally symmetric spaces: p-adic aspects

Joint work with Miaofen Chen and Xu Shen

Grothendieck ICM 70 Nice

Question: $\mathcal{G}$ p-divisible group $/\overline{\mathbb{F}}_p$, (D, φ) = rational Dieudonné module

φ: σ-linear auto.

D: finite dim. $\breve{\mathbb{Q}}_p$-v.s.

$$\breve{\mathbb{Q}}_p = \widehat{\mathbb{Q}_p^{un}} \circlearrowleft \sigma$$

Ex: $\mathcal{G} = A[p^\infty]$ A ab. var. $/\overline{\mathbb{F}}_p$

$$D = H^1_{cris}\left(A / W(\overline{\mathbb{F}}_p)\right)\left[\tfrac{1}{p}\right] \circlearrowleft$$

Cristalline Frobenius

$$\left\{ \begin{array}{c} \text{p-divisible groups}/O_K \text{ isogenous} \\ \text{to } \mathcal{G} \text{ mod } p \end{array} \right\} \xrightarrow{\text{Hodge filtration}} \left\{ \begin{array}{c} \text{Filtrations} \\ \text{of } D_K \text{ of} \\ \text{dim. } \dim \mathcal{G} \end{array} \right\}$$

$K|\breve{\mathbb{Q}}_p$ complete valued

p-adic period mapping analogous to Griffiths period morphism

What is the structure of the image in

$Gr_d(D)$, is it an algebraic variety?

d ← $\dim \mathcal{G}$

Partial answers

Drinfeld:

$$\Omega = \mathbb{P}^{n-1} \setminus \underbrace{\bigcup_{H \in \check{\mathbb{P}}^{n-1}(\mathbb{Q}_p)} H}_{\text{remove a profinite set of alg. var.}} \subset \mathbb{P}^{n-1}$$

rigid analytic open subset

is such a period space for p-div. gp. + particular type add. structure

Serre-Tate-Katz: Ordinary elliptic curves.

$$\mathcal{G} = \mu_{p^\infty} \oplus \mathbb{Q}_p/\mathbb{Z}_p$$

$\mathcal{X} =$ Def. space $\simeq$ Spf $(W(\overline{\mathbb{F}}_p)[[x]])$

period space for ordinary ell. curves

$$\mathcal{X}_\eta \simeq \mathring{\mathbb{B}}^1_{\breve{\mathbb{Q}}_p} \xrightarrow{\text{period map}} \mathbb{A}^1 \subset \mathbb{P}^1$$

$$x \longmapsto \log(1+x)$$

Lubin-Tate - Lafaille - Gross-Hopkins: $\mathcal{G}$ = 1-dim. formal p-div. gp. height $n / \overline{\mathbb{F}}_p$

$\mathcal{X} = \mathrm{Def}(\mathcal{G}) \simeq$ Spf $(W(\overline{\mathbb{F}}_p)[[x_1, \ldots, x_{n-1}]])$

$$\mathcal{X}_\eta \simeq \mathring{\mathbb{B}}^{n-1}_{\breve{\mathbb{Q}}_p} \xrightarrow[\text{étale period mapping}]{\pi} \mathbb{P}^{n-1} = \text{period space}$$

= entire flag manifold

surjective

Fontaine: Characterize the K-points of the period space inside Gr when $[K : \breve{\mathbb{Q}}_p] < +\infty$

$\longrightarrow$ weak admissibility condition

More generally: Rapoport - Zink

$\mathcal{G}/\overline{\mathbb{F}}_p$ + P.E.L. additional structure

polarization endomorphism

Construct: $\mathcal{M} / \mathrm{Spf}\left(W(\overline{\mathbb{F}}_p)\right)$

formal scheme locally

$\mathrm{Spf}\left(W(\overline{\mathbb{F}}_p)[[X_1, \dots, X_n]]\langle Y_1, \dots, Y_m\rangle / \mathrm{Ideal}\right)$

deformation space by quasi-isogenies of $\mathcal{G}$

not only isomorphisms

$\mathcal{M}_\eta$ = Rigid analytic space$/\breve{\mathbb{Q}}_p$ $\qquad V(\mathrm{Ideal}) \hookrightarrow \mathring{\mathbb{B}}^n \times \mathbb{B}^m$

$+ \ \pi : \mathcal{M}_\eta \xrightarrow[\text{period morphism}]{\text{étale}} \mathcal{F}$] flag variety

$\mathcal{F}^a := \mathrm{Im}(\pi)$ open in $\mathcal{F}$ = period space

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weakly admissible

Fontaine's answer: $\mathcal{F}^a \subset \underset{\text{open} = \mathcal{F} \setminus \text{profinite set of Schubert varieties}}{\overset{\text{weakly admissible}}{\mathcal{F}^{wa}}} \subset \mathcal{F}$

open $= \mathcal{F} \setminus$ profinite set of Schubert varieties

s.t. $\mathcal{F}^a(K) = \mathcal{F}^{wa}(K)$ if $[K : \breve{\mathbb{Q}}_p] < +\infty$

But in general: $\mathcal{F}^a \subsetneq \mathcal{F}^{wa}$, for example $\mathcal{F}^a(\mathbb{C}_p) \subsetneq \mathcal{F}^{wa}(\mathbb{C}_p)$

Ex: X Berkovich space $/\breve{\mathbb{Q}}_p$, $x \in X(\mathbb{C}_p) \setminus X(\overline{\breve{\mathbb{Q}}}_p)$

$X \setminus \{x\} \underset{\neq}{\hookrightarrow} X$ but same "classical points"

More generally:

local Shimura datum $\left\{ \begin{array}{l} G/\mathbb{Q}_p \text{ reductive group} \\ \mu : \mathbb{G}_{m\overline{\mathbb{Q}}_p} \longrightarrow G_{\overline{\mathbb{Q}}_p} \text{ \underline{minuscule} cocharacter up to conjugation} \\ [b] \in \underbrace{B(G)}_{\text{Kottwitz set}} = G(\breve{\mathbb{Q}}_p)/\sigma\text{-conjugation} \\ \qquad \text{Kottwitz set} = \{G\text{-isocrystals}\}/\sim \\ \text{s.t. } [b] \in B(G, \mu) \end{array} \right.$

Hodge cocharacter

crystalline Frob.

to be explained later

One can construct (Scholze's diamond + Kedlaya-Liu + the curve)

level structure

$$\underbrace{\mathcal{M}(G,\mu,b)_K}_{\text{local Shimura variety}} \,/\, \breve{E}$$

$\breve{E}$ ← local reflex field

rigid analytic space for $K \subset G(\mathbb{Q}_p)$ compact open

$$+\ \pi : \mathcal{M}(G,\mu,b)_K \xrightarrow[\text{morphism}]{\text{étale period}} \underbrace{\mathcal{F}}_{\text{twisted form of } G/P_\mu}$$

fibers = Hecke orbits

$$\mathcal{F}^a := \mathrm{Im}(\pi) \underset{\text{open}}{\subset} \mathcal{F}.$$

$$\mathcal{F}^a \subset \underbrace{\mathcal{F}^{wa}}$$

$\mathcal{F} \setminus$ profinite set of Schubert varieties

Hartl: has classified all cases when $\mathcal{F}^a = \mathcal{F}^{wa}$ for $G = GL_n$.

any G, not necessarily quasi-split!!

Th (Conjecture of Rapoport and myself):

Suppose b is basic. Then

$\mathcal{F}^{a} = \mathcal{F}^{wa} \iff B(G,\mu)$ is fully HN decomposable

generalization of isoclinic

$\underbrace{\text{isoclinic}}_{\text{one Dieudonné-Manin slope}}$

If Shimura datum $\rightsquigarrow (G,\mu)$ then $\underbrace{B(G,\mu)}_{\text{finite}} \subset B(G)$

classifies the Newton strata of the reduction modulo p of the Shimura variety.

$[b] \in B(G,\mu)$ basic $\longleftrightarrow$ "supersingular" closed stratum

Uniformization $\coprod_i \Gamma_i \backslash \mathcal{M}(G,\mu,b)_K \xrightarrow{\sim}$ tube over supersingular locus

"The Newton polygon of non basic elements in $B(G,\mu)$ touches the Hodge polygon defined by μ outside its extremities"

polygons $\in$ Weyl chamber $\leftrightarrow G$

Ex: $G = SO(n)/\mathbb{Q}_p \leftrightarrow \begin{pmatrix} & & 1 \\ & ⋰ & \\ 1 & & \end{pmatrix} \leftarrow\!\!\!\sim\!\!\sim$ Shimura var. of abelian type associated to $SO(2, n-2)$

quasi-split

$$\mu(z) = \mathrm{diag}(z, 1, \dots, 1, z^{-1})$$

Then $B(G,\mu)$ is fully HN decomposable $\Rightarrow \mathcal{F}^a = \mathcal{F}^{wa}$

$\rightarrow$ for $n = 21$ allows us to compute the p-adic period space of K3 surfaces (polarized) with supersingular reduction

How to prove this theorem? $\longrightarrow$ $\boxed{\text{the curve}}$

$C | \breve{\mathbb{Q}}_p \rightsquigarrow X_C$ the curve = Dedekind scheme $/\mathbb{Q}_p$

alg. closed

$+ \infty \in |X_C| \qquad k(\infty) = C$

There is a construction

$$\{\text{Isocrystals}\} \longrightarrow \{\text{Vector bundles}/X_C\}$$

Can be enhanced to

$$G(\breve{\mathbb{Q}}_p) \longrightarrow \{G\text{-bundles}/X_C\}$$

$$b \longmapsto \mathcal{E}_b$$

Th: $B(G) \xrightarrow{\sim} H^1_{et}(X_C, G)$

$$[b] \longmapsto [\mathcal{E}_b]$$

à la Atiyah-Bott

Moreover b basic $\Leftrightarrow$ $\mathcal{E}_b$ semi-stable

Now for $x \in \mathcal{F}(G,\mu)(C) \rightsquigarrow \mathcal{E}_{b,x}$ = modification at ∞ of $\mathcal{E}_b$ given by x.

uses μ minuscule

Then: $\mathcal{F}(G,\mu)^a(C) = \{x \mid \mathcal{E}_{b,x} \text{ is semi-stable}\}$

* G quasi-split. Fix a Borel subgroup - $\mathcal{E}/X$ G-bundle

$\mathcal{E}$ semi-stable $\iff$ $\forall P$ standard parabolic $\forall \mathcal{E}_P$ reduction of $\mathcal{E}$ to P

$$\forall \chi \in X^*(P/Z_G)^+ \quad \deg(\chi_* \mathcal{E}_P) \leq 0.$$

χ weakly admissible $\iff$ $\forall P$ $\forall b_P$ $\underbrace{\text{reduction of } b \text{ to } P}_{\text{sub-isocrystal}}$

$\rightsquigarrow \mathcal{E}_{b_P}$ reduction of $\mathcal{E}_b$ to P

$\rightsquigarrow (\mathcal{E}_{b,x})_P$ reduction of $\mathcal{E}_{b,x}$ to P

reductions to parabolics transfert via modifications

$$\forall \chi \in X^*(P/Z_G)^+ \quad \deg(\chi_* (\mathcal{E}_{b,x})_P) \leq 0$$

admissible: test on all reductions of $\mathcal{E}_{b,x}$ to P

weakly " : test on reductions coming from a reduction of b